Larmor Precession Tutorial

Here is instructions on how to use the simulation program.(The first simulation is used in question 5)

1. *Double click the file "osp_spins"*
2. *On the top of the new window, click "File→Open Internal"*
3. *Choose the second file "osp_spins_about.xset"*
4. *On the left column of the new window, double click "OSP Spins Program"*
5. *Double click "custom configuratin"*
6. *In the new window, click "File→Load XML" , and then choose an xml file to open the corresponding simulation.*

Initial state $\left|\uparrow\right\rangle_z$ *, measure* $\hat{S}_z$ *(Question 1~7)*

1. The Hamiltonian of a spin-1/2 particle in a magnetic field $\vec{B} = B_0\hat{z}$ is $\hat{H} = -\gamma B_0 \hat{S}_z$, where $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the z-component of spin angular momentum. Which of the following is the eigenvalue of the Hamiltonian $\hat{H} = -\gamma B_0 \hat{S}_z$?

A. $E_n = \mp \gamma B_0$

B. $E_n = \mp \frac{\gamma B_0}{2}$

C. $E_n = \mp \frac{\hbar}{2}\gamma B_0$

D. $E_n = \mp \frac{\hbar}{2}$

2. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $\left|\chi(0)\right\rangle = \left|\uparrow\right\rangle_z$. Which of the following equation correctly represents the state $\left|\chi(t)\right\rangle$ of the electron? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

A. $\left|\chi(t)\right\rangle = \left|\uparrow\right\rangle_z$

B. $\left|\chi(t)\right\rangle = e^{i\gamma B_0 t/2} \cdot \left|\uparrow\right\rangle_z$

C. $\left|\chi(t)\right\rangle = e^{i\gamma B_0 t/2} \cdot \left|\uparrow\right\rangle_z + e^{-i\gamma B_0 t/2} \cdot \left|\downarrow\right\rangle_z$

D. $\left|\chi(t)\right\rangle = ae^{i\gamma B_0 t/2} \cdot \left|\uparrow\right\rangle_z + be^{-i\gamma B_0 t/2} \cdot \left|\downarrow\right\rangle_z$

3. Consider the following conversation between Andy and Caroline about the above Hamiltonian operator.

Andy: $\hat{H}$ is essentially $\hat{S}_z$ except for some multiplicative constants. Therefore, the eigenstates of $\hat{S}_z$ will also be the eigenstates of $\hat{H}$.

Caroline: No. The presence of magnetic field will make the eigenstates of $\hat{S}_z$ and $\hat{H}$ different. The eigenstates of $\hat{H}$ will change with time in a non-trivial manner.

Andy: I disagree. If the magnetic field had a time dependence, e.g., $B = B_0 \cos(\omega t)\hat{k}$, the eigenstates of $\hat{H}$ will change with time in a non-trivial manner but not for the present case where $\vec{B}$ is constant.

With whom do you agree?

A. Andy
B. Caroline
C. Neither

4. In the previous problem (problem 2), suppose t_0 satisfies $-\gamma B_0 t_0 / 2 = \pi / 6$. What is the probability of getting $\left|\uparrow\right\rangle_z$ when we measure $\hat{S}_z$ at $t = t_0$?

5. **Now you can use "File→Load XML→ larmor_z-up_Sz_30" to check your answer to the previous problem (problem 4). In the simulation, the red block is the magnetic field, and the blue block is the Stern-Gerlach apparatus to measure $\hat{S}_z$. When the upper detector after SGZ clicks, the state of the particle measured is $\left|\uparrow\right\rangle_z$. The number "30" in the magnetic field means that the particle will stay in the magnetic field for time t_0 satisfying $-\gamma B_0 t_0 / 2 = \pi / 6 = 30^\circ$**. Explain whether what you observed in the simulation is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

6. In the previous problem (problem 2), does the probability of getting $|\uparrow\rangle_z$ depend on the time t? Explain.

Now you can use the simulations "larmor_z-up_Sz_30", "larmor_z-up_Sz_45" and "larmor_z-up_Sz_90" to check your answer. In these three simulation, the time t_0 satisfies $-\gamma B_0 t_0 / 2 = \pi / 6 = 30^\circ$, $-\gamma B_0 t_0 / 2 = \pi / 4 = 45^\circ$ and $-\gamma B_0 t_0 / 2 = \pi / 2 = 90^\circ$ respectively.

7. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

Initial state $|\uparrow\rangle_z$, measure $\hat{S}_x$ (Question 8~15)

8. An electron in a magnetic field $\vec{B} = B_0 \hat{z}$ is initially in the state $|\chi(0)\rangle = |\uparrow\rangle_z$. After time t, the electron will be in state $|\chi(t)\rangle$. How can you express $|\chi(t)\rangle$ in the spin-x basis $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$?

9. In the previous problem (problem 8), suppose t_0 satisfies that $-\gamma B_0 t_0 / 2 = \pi / 6$, what is the probability of getting $|\uparrow\rangle_x$ when we measure the observable S_x at $t = t_0$?

10. **Now you can use the simulations "larmor_z-up_Sx_30" to check your answer.** Explain whether what you observed in the simulation is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

11. In the previous problem (problem 8), does the probability of getting $|\uparrow\rangle_x$ depend on the time t ?

Now you can use the simulations "larmor_z-up_Sx_30" , "larmor_z-up_Sx_45" and "larmor_z-up_Sx_90" to check your answer. In these three simulation, the time t_0 satisfies $-\gamma B_0 t_0 / 2 = \pi / 6 = 30^\circ$, $-\gamma B_0 t_0 / 2 = \pi / 4 = 45^\circ$ and $-\gamma B_0 t_0 / 2 = \pi / 2 = 90^\circ$ respectively.

12. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

13. Calculate the expectation value of $\hat{S}_x$ at time t. Then evaluate $\langle \hat{S}_x \rangle$ at time t_0 satisfying $-\gamma B_0 t_0 / 2 = \pi / 6 = 30^\circ$.

14. Find the expectation value by counting the number of clicks in the upper and lower detectors in the simulation **"larmor_z-up_Sx_30"**? Is this simulation result the same as what you calculated in the previous question (problem 13)?

15. If you had to explain to a friend what expectation value physically means and why the calculated value and the value from the simulation are the same, what would you say?

Initial state $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, *measure* $\hat{S}_z$ *(Question 16~23)*

16. If the state of the system at an initial time t = 0 is given by $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, which one of the following is the state, $|\chi(t)\rangle$, after a time t? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

A. $|\chi(0)\rangle = e^{i\gamma B_0 t/2} \cdot \left(a|\uparrow\rangle_z + b|\downarrow\rangle_z\right)$

B. $|\chi(0)\rangle = e^{-i\gamma B_0 t/2} \cdot \left(a|\uparrow\rangle_z + b|\downarrow\rangle_z\right)$

C. $|\chi(0)\rangle = e^{i\gamma B_0 t/2} \cdot \left((a+b)|\uparrow\rangle_z + (a-b)|\downarrow\rangle_z\right)$

D. $|\chi(0)\rangle = ae^{i\gamma B_0 t/2} \cdot |\uparrow\rangle_z + be^{-i\gamma B_0 t/2} \cdot |\downarrow\rangle_z$

17. In problem 16, which one of the following gives the correct outcomes when we measure $\hat{S}_z$ at $t = t_0$? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

A. $\hbar/2$ with a probability $ae^{i\gamma B_0 t/2}$ and $-\hbar/2$ with a probability $be^{-i\gamma B_0 t/2}$

B. $\hbar/2$ with a probability $a^2 e^{i\gamma B_0 t/2}$ and $-\hbar/2$ with a probability $b^2 e^{-i\gamma B_0 t/2}$

C. $\hbar/2$ with a probability $|a^2|$ and $-\hbar/2$ with a probability $|b^2|$

D. $\hbar/2$ and $-\hbar/2$ with equal probability

18. Consider the following conversation between Andy and Caroline about measuring $\hat{S}_z$ in the state $|\chi(t)\rangle$

Andy: Since the probability of measuring $\hbar/2$ is $|a^2|$, $-\hbar/2$ is $|b^2|$ and $|a^2| + |b^2| = 1$, we can choose our a and b as $a = e^{i\phi_1}\cos(\alpha)$ and $b = e^{i\phi_2}\sin(\alpha)$ where α , ϕ_1 and ϕ_2 are free parameters.

Caroline: I agree. Since $\cos^2(\alpha) + \sin^2(\alpha) = 1$, it gives the same relation as $|a^2| + |b^2| = 1$ and there is no loss of generality.

Do you agree with Andy and Caroline? If yes, for the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$, what are the values of a and b when you express it in terms of $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$? And what are the values of α , ϕ_1 and ϕ_2 when you express it in terms of α defined earlier as $a = e^{i\phi_1}\cos(\alpha)$ and $b = e^{i\phi_2}\sin(\alpha)$.

19. Suppose the initial state is $|\chi(0)\rangle = \cos\alpha|\uparrow\rangle_z + \sin\alpha|\downarrow\rangle_z$, which one of the following is the expectation value $\langle \hat{S}_z \rangle = \langle \chi(t)|\hat{S}_z|\chi(t)\rangle$? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

A. $\cos(2\alpha)\cos(\gamma B_0 t)\hbar/2$

B. $\sin(2\alpha)\sin(\gamma B_0 t)\hbar/2$

C. $\cos(2\alpha)\hbar/2$

D. $\sin(2\alpha)\hbar/2$

Now you can check your answers of the previous problems with a concrete example below (problem 20 to 23).

20. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$. Suppose t_0 satisfies that $-\gamma B_0 t_0/2 = \pi/6$. What is the probability of getting $|\uparrow\rangle_z$ when we measure the observable S_z at $t = t_0$? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

Now you can use "File→Load XML→ larmor_z-up+z-down_Sz_30" to check your answer.

21. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

22. In problem 21, does the probability of getting $|\uparrow\rangle_z$ depend on the time t? What is the expectation value of $\hat{S}_z$?

Now you can use “larmor_z-up+z-down_Sz_30”, “larmor_z-up+z-down_Sz_45” and “larmor_z-up+z-down_Sz_90” to check your answer. In these three simulation, the time t_0 satisfies $-\gamma B_0 t_0/2=\pi/6=30^\circ$, $-\gamma B_0 t_0/2=\pi/4=45^\circ$ and $-\gamma B_0 t_0/2=\pi/2=90^\circ$ respectively.

23. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

Initial state $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, *measure* $\hat{S}_x$ *(Question 24~29)*

Now let’s go back to the most general initial state at time t = 0 which is given by $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$***. The Hamiltonian is*** $\hat{H} = -\gamma B_0 \hat{S}_z$***. Answer problem 26 and 27.***

24. Find the state $|\chi(t)\rangle$ at time t in the spin-x basis $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$. Express your answer in terms of α defined earlier as $a = e^{i\phi_1}\cos(\alpha)$ and $b = e^{i\phi_2}\cos(\alpha)$.

25. Suppose the initial state is $|\chi(0)\rangle = \cos\alpha|\uparrow\rangle_z + \sin\alpha|\downarrow\rangle_z$, which one of the following is the expectation value $\langle \hat{S}_x\rangle = \langle\chi(t)|\hat{S}_x|\chi(t)\rangle$? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

A. $\cos(2\alpha)\cos(\gamma B_0 t)\hbar/2$

B. $\sin(2\alpha)\cos(\gamma B_0 t)\hbar/2$

C. $\cos(2\alpha)\hbar/2$

D. $\sin(2\alpha)\hbar/2$

Now you can check your answers of the previous problems with a concrete example in which $\phi_1 = 0$ **,** $\phi_2 = 0$**,** $\alpha = \pi/4$ ***(problem 26 to 29).***

26. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$.

    The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$. What is the probability of getting $|\uparrow\rangle_x$ at time t?

    Does the probability of getting $|\uparrow\rangle_x$ depend on the time t? Explain.

Now you can use "larmor_z-up+z-down_Sx_30" , "larmor_z-up+z-down_Sx_45" and "larmor_z-up+z-down_Sx_90"to check your answer. In these three simulation, the time t_0 **satisfies** $-\gamma B_0 t_0/2 = \pi/6 = 30^\circ$**,** $-\gamma B_0 t_0/2 = \pi/4 = 45^\circ$ **and** $-\gamma B_0 t_0/2 = \pi/2 = 90^\circ$ **respectively.**

27. In problem 26, the initial state is just $|\chi(0)\rangle = |\uparrow\rangle_x$, but why the probability of getting $|\uparrow\rangle_x$ is time-dependent? Can we write the state $|\chi(t)\rangle$ at time t as $|\chi(t)\rangle = e^{i\gamma B_0 t/2} \cdot |\uparrow\rangle_x$?

    Explain.

28. In problem 26, what is the expectation value of $\hat{S}_x$ at time t? Does $\langle \hat{S}_x \rangle$ depend on time?

    When t_0 satisfies that $-\gamma B_0 t_0/2 = \pi/6$, what is $\langle \hat{S}_x \rangle$ at $t = t_0$?

Now you can use "larmor_z-up+z-down_Sx_30 " to check your answer.

29. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question.

Initial state $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, measure $\hat{S}_y$ and find the general principal of time-dependence in Larmor precession.(Question 30~39)

30. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$. The Hamiltonian operator is $\hat{H} = -\gamma B_0\hat{S}_z$. What is the probability of getting $|\uparrow\rangle_y$ if we measure $\hat{S}_y$ at $t = t_2$? t_2 satisfies that $-\gamma B_0 t_2 / 2 = \pi / 4$

Now you can use "File→Load XML→ larmor_z-up+z-down_Sy " to check your answer.

31. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question.

32. Explain in your own words why $\langle\hat{S}_z\rangle$ above does not depend on time where as $\langle\hat{S}_x\rangle$ and $\langle\hat{S}_y\rangle$ do.

33. Consider the following conversation between Pria and Mira about $\langle \hat{S}_z \rangle$ in state $|\chi(t)\rangle$:

Pria: Since the state of the system $|\chi(t)\rangle$ evolves in time, the expectation value $\langle \hat{S}_z \rangle$ will depend on time.

Mira: I disagree with the second part of your statement. The time development of the expectation value of any operator $\hat{A}$ is given by $\frac{d\langle \hat{A} \rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$. In our case, $[\hat{H}, \hat{S}_z] = 0$ and the operator $\hat{S}_z$ does not have any explicit time dependence so $\frac{\partial \hat{S}_z}{\partial t} = 0$.

Thus, $\frac{d\langle \hat{S}_z \rangle}{dt} = 0$ and the expectation value will not change with time.

With whom do you agree?

In problem 34~39, the Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

34. Choose all of the following statements that are true about the expectation value $\langle \vec{S} \rangle$ in the state $|\chi(t)\rangle$:

A. $\langle \vec{S} \rangle$ will always depend on time because $[\hat{H}, \hat{S}] \neq 0$

B. $\langle \vec{S} \rangle$ cannot depend on time because the expectation value of an observable is its time-averaged value.

C. $\langle \vec{S} \rangle$ is time-independent only when the initial state is purely $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$

35. Choose all of the following statements that are true about the expectation value $\langle \vec{S} \rangle$ in the state $|\chi(t)\rangle$ when the initial state is not purely $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$:

(1) The z component of $\langle \vec{S} \rangle$, i.e., $\langle \hat{S}_z \rangle$, is time-independent.

(2) The x and y components of $\langle \vec{S} \rangle$ change with time. When $\langle \hat{S}_x \rangle$ is maximum $\langle \hat{S}_y \rangle$ is a minimum and vice versa.

(3) The magnitude of the maximum values of $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ are the same.

A. 1 and 2
B. 1 and 3
C. 2 and 3
D. All of the above

36. Choose all of the following statements that are true about the expectation value $\langle \vec{S} \rangle$ in the state $|\chi(t)\rangle$:

(1) The vector $\langle \vec{S} \rangle$ can be thought to be precessing about the z axis at an angle 2α.

(2) The vector $\langle \vec{S} \rangle$ can be thought to be precessing about the z axis with a frequency $\omega = \gamma B_0$.

(3) All the three components of vector $\langle \vec{S} \rangle$ change as it precesses about the z axis.

A. 1 and 2
B. 1 and 3
C. 2 and 3
D. All of the above

37. Choose a three dimensional coordinate system in the spin space with the z axis in the vertical direction. Draw a sketch showing the precession of $\langle \vec{S} \rangle$ about the z axis when the state of the system starts out in $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$. Show the angle that $\langle \vec{S} \rangle$ makes with the z axis and the precession frequency explicitly.

Show the projection of $\langle \vec{S} \rangle$ along the x, y and z axes at two separate times. Explain in words why the projection of $\langle \vec{S} \rangle$ along the z direction does not change with time but those along the x and y directions change with time.

38. Explain in your own words, why for $\hat{H} = -\gamma B_0 \hat{S}_z$, if the initial state is $|\uparrow\rangle_z$, $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$ and $\langle \hat{S}_z \rangle$ are all time independent. But if the initial state is $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$, then $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ depend on time while $\langle \hat{S}_z \rangle$ does not. Your explanation should bring out general principle of when expectation value of an operator will not depend on time.